Lesson

2-5 Exponential Models

Vocabulary

exponential regression

half-life

BIG IDEA Exponential models are used in many fields in which data follow some kind of natural growth or decay. Exponential regression is used to fit exponential models to data.

Finding an Exponential Function Using a System of Equations

Exponential models describe situations in diverse fields such as biology, paleontology, sociology, physics, and economics.

Populations very often grow at a constant rate, at least in the short run. Therefore, it is natural to fit an exponential model to population data. The average population growth rate in the U.S. is about 0.883%, but in areas that are growing quickly, the rate can be much higher.

Mental Math

Veronica is preparing a study plan for a biology test. She plans to study for four days, and each day she plans to study 50% longer than the previous day. On the first day she plans to study for 24 minutes. How many hours and minutes total will she study?

Example 1

Huntley, Illinois had been a small farming town. But when a large housing development was built, the population growth pattern changed. Two special censuses gave village planners the data in the table at the right.

Year	Population
2003	12,270
2005	16,719

Source: The Village of Huntley

a. Find an exponential model for the data. Let $p(t)$ be the population t years after 2000.

b. Predict the population of Huntley in the year 2015.

Huntley, IL

Solution

a. A general equation for the model is $p(t) = ab^t$. From the table, $p(3) = 12{,}270$ and $p(5) = 16{,}719$. Substitute these values into the general equation to get a system.

$$\begin{cases} 12{,}270 = ab^3 \\ 16{,}719 = ab^5 \end{cases}$$

Divide the second equation by the first. This can be done because $a \neq 0$ and $b \neq 0$.

$$\frac{16{,}719}{12{,}270} = b^2$$

Because the base b must be positive, take the positive square root.

$$b = \sqrt{\frac{16{,}719}{12{,}270}} \approx 1.1673$$

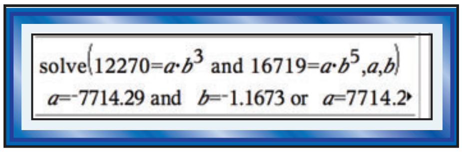
This is the growth factor. The population was growing about 17% per year.

To find a, substitute this value of b into one of the two equations in the system. Using the first equation,

$$12{,}270 = a(1.1673)^3$$
$$a \approx \frac{12{,}270}{(1.1673)^3} \approx 7714.$$

According to this model, the initial population of Huntley in 2000 was 7714. The full exponential model is $p(t) = 7714(1.1673)^t$.

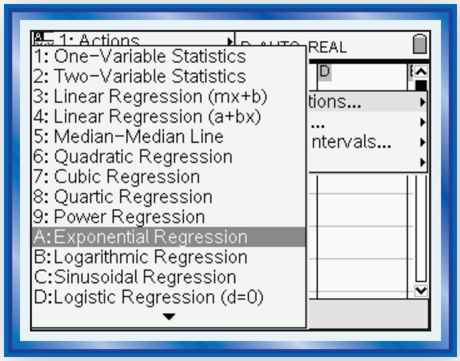
Check Use a CAS to solve the system of equations. A partial solution is shown at the right. The negative answers are not valid in this context.



b. The model predicts the population in 2015 to be
$p(15) = a \cdot b^{15} = 7714(1.1673)^{15} \approx 78{,}500.$
Actually, demographers expect this growth rate to slow before then, but the model describes what would happen based on the two special census years.

Exponential Regression

The model for the population of Huntley was derived from just two data points. Two points are enough to algebraically determine an exponential function that is a perfect fit for those two points. If there are more than two data points, there may not be a model that fits all the data perfectly. However, if many data points follow an approximately exponential pattern, a statistics utility will find an **exponential regression** curve that models the pattern well.



GUIDED

Example 2

Bald eagles were once threatened with extinction. In the 48 contiguous states, their numbers were at an all-time low of 417 in 1963. But protection programs helped them rebound. In 2007, they were removed from the list of endangered species kept by the U.S. Fish and Wildlife Services.

a. Use a statistics utility to fit an exponential model of the form $f(x) = ab^x$ to the data. Let x be "years after 1960." Report the values of a and b in the table on the next page to the nearest thousandth.

b. Superimpose the graph of the exponential model on the scatterplot and describe how well the exponential curve fits the data.

c. Identify the initial amount and the growth factor and explain their meanings.

d. Find the residuals for the model's predicted values for 2000 and 2005.

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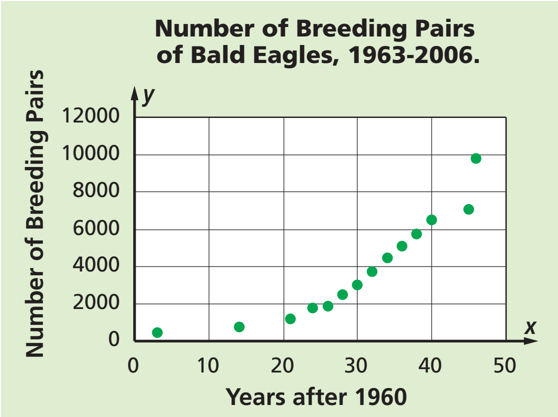
Bald eagles have a wing span of 6 to 8 feet.



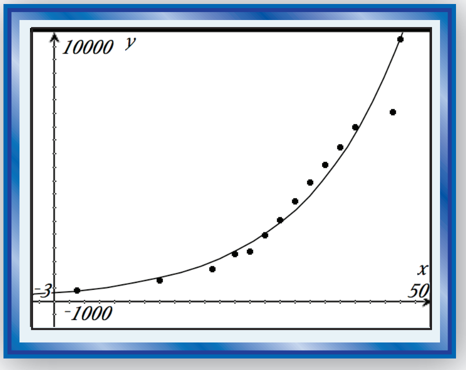
Year	Number of Breeding Pairs
1963	417
1974	791
1981	1188
1984	1757
1986	1875
1988	2475
1990	3035
1992	3749
1994	4449
1996	5094
1998	5748
2000	6471
2005	7066
2006	9789

Source: U.S. Fish and Wildlife Services

Solution

a. A statistics utility gives a ≈ 296.177 and b ≈ 1.079. Therefore, f(x) = __?__.

b. Your graph should look similar to the screenshot at the right. The model is close to most of the points of the scatterplot.



c. The initial value of the exponential model is about __?__ breeding pairs, which corresponds to the year __?__. The growth factor is __?__. This means that during the years 1963 through 2006, the eagle population had a growth rate of about __?__ % per year.

d. In 2000, the predicted number of breeding pairs is f(40) ≈ __?__ pairs. So for 2000, residual = observed value − predicted value = __?__ − __?__ = __?__. In 2005, the predicted number of breeding pairs is ≈ __?__ pairs. For 2005, error = __?__ − __?__ = __?__.

Letting x = years after 1960 makes a difference in the model. If the actual year is used as the independent variable, the statistics utility model is $f(\text{year}) = (9.181 \cdot 10^{-63}) \cdot (1.079)^{\text{year}}$. The exponents are so large that a small change in the growth factor due to rounding, for example, would create a large difference in the predicted values. This is a consideration for exponential models. However, with linear and quadratic models, it often does not make a difference whether you use the year as it is or use the number of years from a given point in time.

STOP QY

QY

a. Without a calculator, estimate the value of $1.077^{1963} - 1.076^{1963}$.

b. Estimate the value in Part a with a calculator.

c. How close was your estimate to the actual value?

Half-Life and Exponential Decay

Radioactive elements are useful in situations involving detective work, such as diagnosing health problems with barium x-rays or finding the age of archeological artifacts with carbon dating.

The **half-life** of a radioactive element is the amount of time it takes an original quantity to decay to half that amount. If you know the half-life of a radioactive element and the amount of the substance at one point in time, you can find the original amount.

In 2007, the element polonium was in the news when London police detectives investigated the poisoning of former Russian KGB agent Alexander Litvinenko. Since polonium had never been known to be used in a poisoning, the authorities did not look for evidence of it until weeks after the crime had taken place. As a consequence, they had to work backwards from the evidence to calculate the amount of polonium used on the victim. They made use of the fact that the half-life of polonium is 138 days.

Example 3

Detectives in the Litvinenko investigation found polonium on a cup in a hotel that he had visited. Suppose that 4 micrograms were found, and it had been 30 days since Litvinenko was there.

a. Find how much polonium was on the cup originally.

b. Derive a model for this situation.

Solution

a. Let t represent the number of days since a micrograms of polonium were placed on the cup. Then $f(t) = ab^t$ is the amount of polonium remaining. First, find the daily decay factor b. The half-life tells us that it takes 138 days for a micrograms of polonium to decay to $\frac{1}{2}a$ micrograms. So when $t = 138$, we know that $f(t) = \frac{1}{2}a$. Substitute these values into $f(t) = ab^t$.

$$\frac{1}{2}a = ab^{138}$$

$$\frac{1}{2} = b^{138} \qquad \text{Divide both sides by } a.$$

$$\left(\frac{1}{2}\right)^{\frac{1}{138}} = \left(b^{138}\right)^{\frac{1}{138}} \qquad \text{Raise each side to the } \tfrac{1}{138}\text{th power.}$$

$$b \approx 0.995$$

The decay factor is about 0.995. The polonium decay function is $f(t) = a(0.995)^t$. Now use the information that there were 4 micrograms after 30 days. Again, substitute.

$$4 \approx a(0.995)^{30}$$

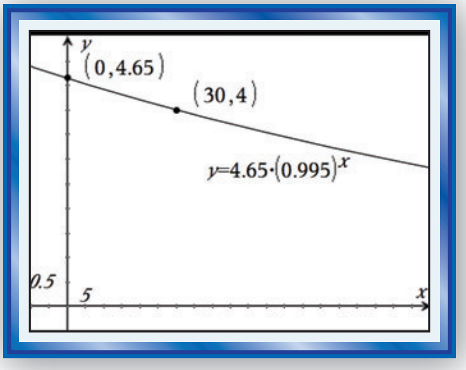
$$a \approx 4.649$$

There were about 4.65 micrograms of polonium originally.

b. Substitute the values of a and b into $f(t) = ab^t$.
After t days, there are $f(t) = 4.65(0.995)^t$ micrograms of polonium remaining.

Check

Graph $f(x) = 4.65(0.995)^t$. Trace to see that (0, 4.65) and (30, 4.00) are on the graph.



Questions

COVERING THE IDEAS

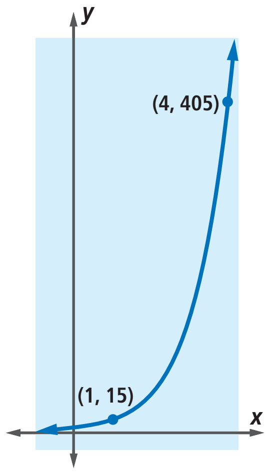
1. Suppose an exponential function with equation $f(t) = ab^t$ contains the two points (3, 20) and (10, 156).

a. Write the system of equations that results from substituting the two points into the equation.

b. Solve the system to yield an equation for the function. Round your values for a and b to the nearest thousandth.

c. Check your equation for the two points (3, 20) and (10, 156).

2. The graph at the right shows two ordered pairs that lie on the graph of an exponential function. Find an equation for the exponential function that contains the two points.



3. In a study of the change in an insect population, there were about 170 insects four weeks after the study began, and about 320 after two more weeks. Assume an exponential model of growth.

a. Find an equation relating the population to the number of weeks after the study began.

b. Estimate the initial number of insects.

c. Predict the number of insects five weeks after the study began.

4. The prices of some diamonds of different sizes are given in the table at the right.

a. Find an exponential regression model for this data.

b. According to your model, what would be the price of a 2-carat diamond?

Weight (carats)	Price
0.25	$504
0.40	$1,040
0.55	$1,925
0.80	$3,680
1.25	$10,000
1.65	$18,150

5. The half-life of barium, which is used in CAT scans for medical diagnosis, is 2.6 minutes. Suppose a patient swallows a drink containing 10 units of barium prior to getting a CAT scan.

a. Find an exponential model for the amount of barium left in the patient's system as a function of the number of minutes that have passed since drinking the barium.

b. Find the amount of barium left after an hour.

6. Safety engineers monitor workplaces to see that workers are not exposed to unsafe levels of hazardous chemicals. Suppose that one chemical has a half-life of 7 days. If a worker currently has 18 units of this chemical in his body and was exposed 5 days ago, how much was the initial dose in his body?

7. The table at the right shows how the number of U.S. cell-phone subscribers has grown since 1985.

   a. Create an exponential regression model for the number of subscribers t years after 1985. Report your value of a rounded to the nearest integer and b rounded to the nearest thousandth.

   b. Describe how well the model fits the data.

   c. Determine the year in which the value predicted by the model differs by the greatest *percent* from the actual value.

Year	Number of Cell-Phone Subscribers (thousands)
1985	340
1987	1,231
1989	3,509
1991	7,557
1993	16,009
1995	33,786
1997	55,312
1999	86,047
2001	128,375
2003	158,722

Source: CTIA - The Wireless Association

APPLYING THE MATHEMATICS

8. Benjamin Franklin specified in his will that "1000 pounds sterling" were to be given to the town of Boston for the purposes of providing loans at interest to apprentices. He expected the loans to be repaid and unused money to be well invested. He predicted, "If this plan is executed, and succeeds as projected without interruption for one hundred years, the sum will then be one hundred and thirty-one thousand pounds…." What annual yield did Franklin expect on his gift to Boston?

9. Radium has a half-life of 1620 years. Suppose 3 g of radium is present initially.

   a. Complete the table below for this situation.

Number of Half-Lives	0	1	2	3
t = Number of Years After Start	0	1620		
$f(t)$ = Amount of Radium Present (grams)	3			

   b. Give an exponential model for the amount of radium left as a function of time t.

   c. How much radium would you expect to find after 4000 years?

10. A tour guide noticed that larger groups took more time to assemble for an event. The guide collected the data below.

Number of People	2	3	4	5	6	7	8	9	10
Time to Assemble (minutes)	2	2.6	3.4	4.4	5.7	7.4	9.7	12.5	16.3

   a. Make a scatterplot of the data.

   b. Find the linear regression model and graph it.

   c. Find the exponential regression model and graph it.

   d. Which of the two models seems to fit the data better? Why?

REVIEW

11. Consider $f(x) = 7^x$ and $g(x) = 8^x$. Without graphing, which function has greater values when:
   a. $x > 0$.
   b. $x < 0$ (Lesson 2-4)

In 12 and 13, Fill in the Blank. (Lesson 2-4)

12. All functions with equations of the form $y = a \cdot b^x$ contain the point ___?___.

13. The graph of every function in the form $y = b^x$ where $b > 0$ and $b \neq 1$ is always above the ___?___.

14. Use the Huntley data from Example 1. (Lesson 2-2)
   a. Find a linear model determined by the data for 2003 and 2005.
   b. What does the linear model predict for the 2015 population?

In 15 and 16, give an example of a function satisfying the given condition. In each case, give a domain and a rule. (Lesson 2-1)

15. The range is the set of all negative real numbers.

16. The range is the set of all nonnegative real numbers.

17. In 2004, of the 50 countries in the Western Hemisphere that consume petroleum, the mean consumption was about 608 thousand barrels per day per country. (Lesson 1-2)
   a. Find the total number of barrels consumed per day by all 50 countries.
   b. The United States consumed 20,731 thousand barrels per day in 2004. What would the mean be if the U.S. was not included?
   c. What percentage of the Western Hemisphere consumption is due to the U.S. consumption?
   d. Explain why mean consumption for all 50 countries is misleading given the U.S. consumption.

18. Consider an isosceles right triangle. (Previous Course)
   a. If the legs have length s, how long is the hypotenuse?
   b. If the hypotenuse has length h, how long are the legs?

EXPLORATION

19. Find out what happened to Alexander Litvinenko.

Over a barrel?

In 2007, the U.S. imported 3,437,000 barrels of petroleum per day.

QY ANSWER

a. Answers vary.

b. about $1.455 \cdot 10^{63}$, a number with 64 digits in base 10

c. Most people estimate a much smaller difference.